A Framework for Analyzing Geometric Pattern Tasks

Teachers can use geometric patterns to promote students' understanding of functional relationships.

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There is growing recognition that learning to reason algebraically is an essential component of middle-grades mathematics education. But what does it mean to reason algebraically? Blanton and Kaput (2005) describe it as a "process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (p. 413). Algebraic reasoning takes various forms, including functional thinking in which students explore generalizing patterns to describe functional relationships (Blanton and Kaput 2005).

Principles and Standards for School Mathematics (NCTM 2000) states that focusing on understanding patterns, relations, and functions is a primary goal of algebra instruction. Relationships that are inherent in numerical and geometric (visual) patterns can be represented using words, tables, graphs, and symbols. Students can make and explain generalizations about patterns and use those relationships to make predictions.

Driscoll (1999) highlights the importance of building rules to represent functions: "Critical to algebraic thinking is the capacity to recognize patterns and reorganize data to represent situations in which input is related to output by well-defined functional rules" (p. 2). Working with patterns involves exploring and expressing regularities.

In the primary grades, students explore patterns that repeat. In the middle-grades curriculum, patterns that grow have a larger presence. The real power in using the latter form of

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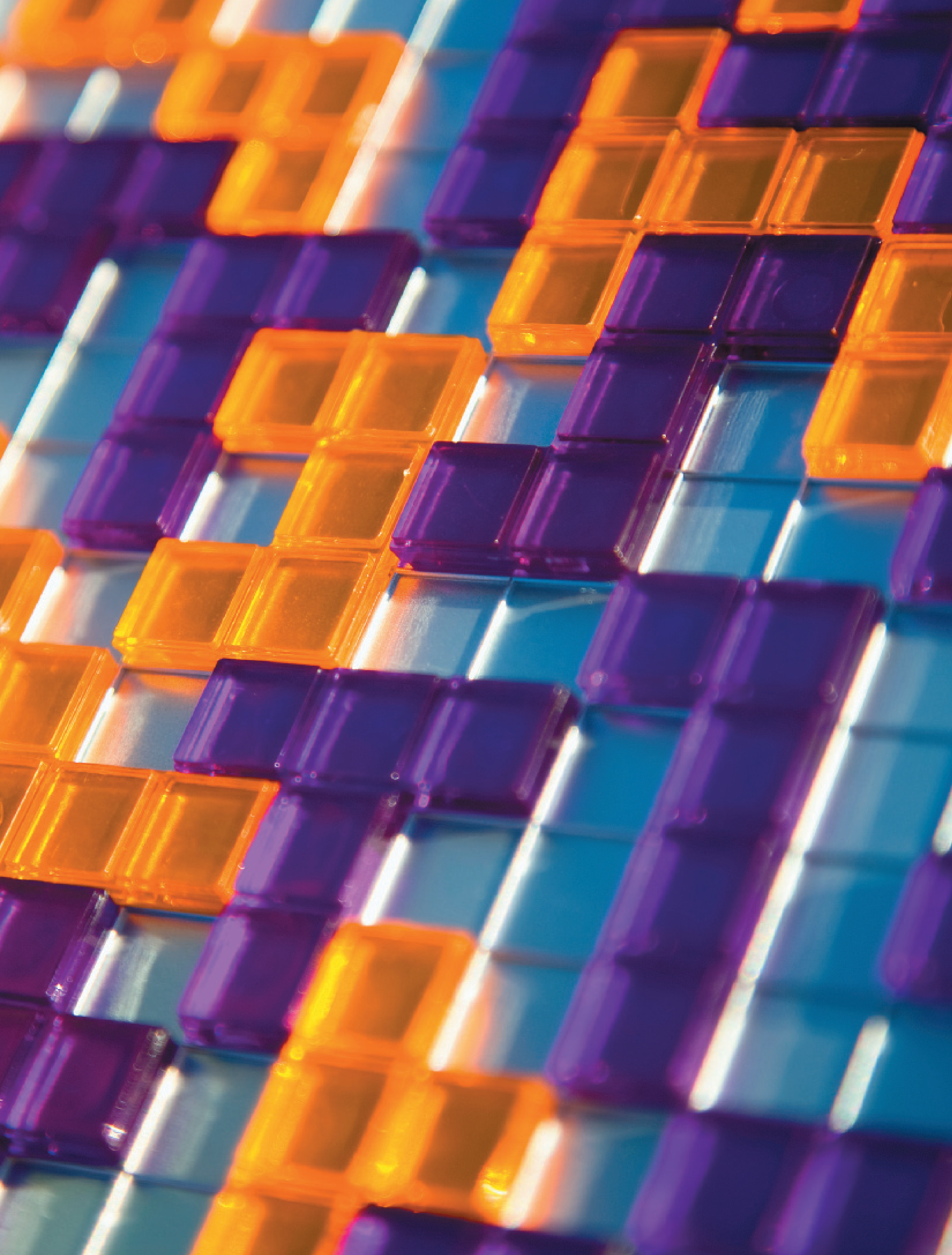
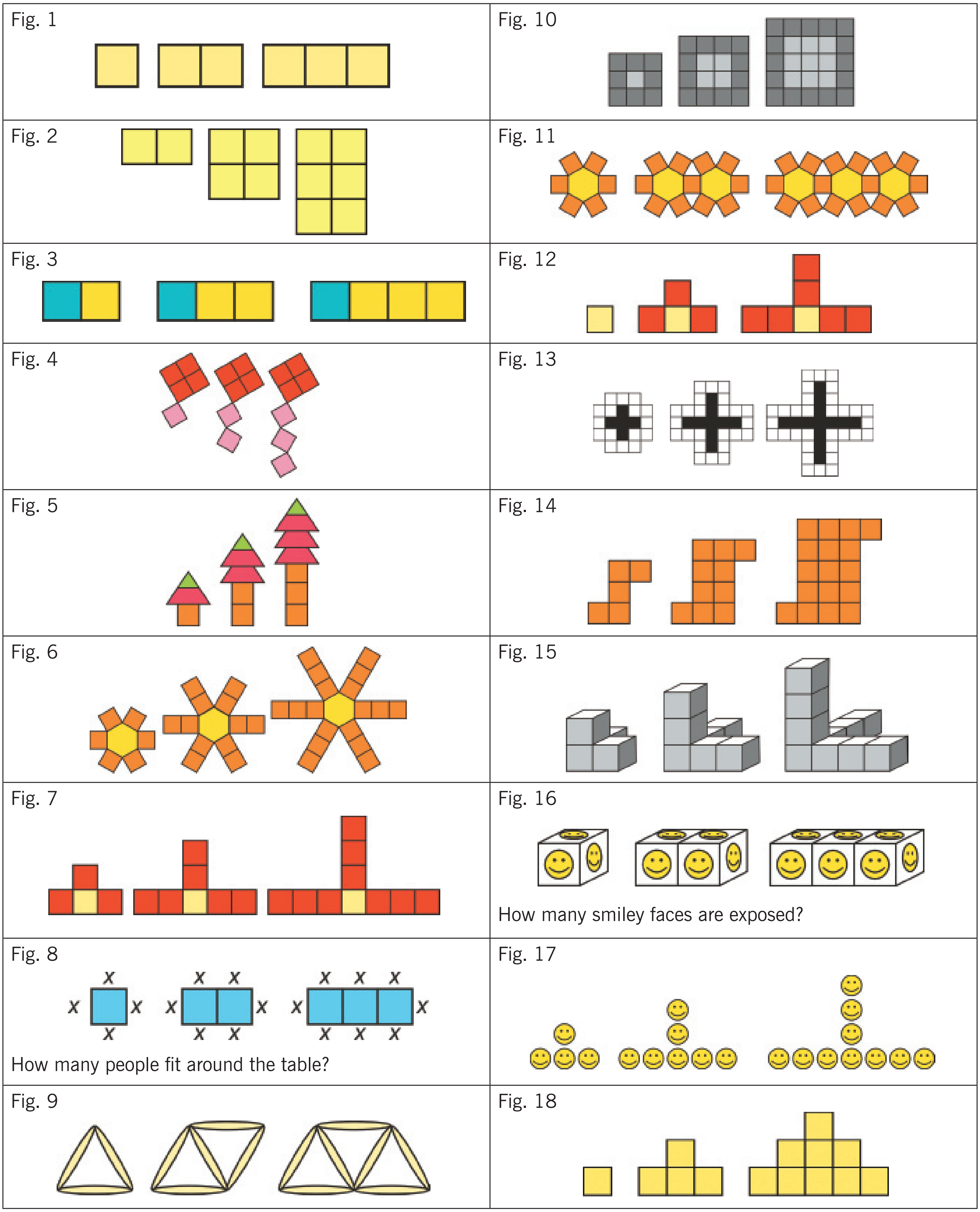




Figs. 1–18 Examples of geometric pattern tasks

these tasks is found in relating numbers to growing pattern tasks that are visual (e.g., made with geometric figures) by asking students to discover the regularities involved and develop generalizations for function rules.

In our investigations of growing pattern tasks, we have explored the development of a framework that can be used to characterize the nature and complexity of such tasks. We are particularly interested in visual growing patterns, referred to as *geometric patterns* (NCTM 2000) or *pictorial growth patterns* (Billings, Tiedt, and Slater 2007/2008), which involve the use of figural objects (Rivera 2007).

The term *figural objects* refers to items that possess both spatial properties and conceptual qualities (Rivera 2007, p. 69). *Figural* means that the objects (or pictures of the same) "possess attributes or exhibit relationships among one another" (Rivera and Becker 2005, p. 199).

In exploring geometric patterns (see **figs. 1–18**), the focus is on the use of inductive reasoning to analyze sequences of figural and numerical cues, with numerical cues following a certain numerical order (Rivera and Becker 2005). In particular, we emphasize the use of figural reasoning during the process of inductive reasoning. "A numerical mode of inductive reasoning uses algebraic concepts and operations (such as finite differences), whereas a figural mode relies on relationships that could be drawn visually from a given set of particular instances" (Rivera and Becker 2005, p. 199).

When students use figural reasoning, they are able to make sense of patterns, such as those in **figure 1**, by paying attention to visual cues that can be organized and translated to numeric sequences. These cues explain and support pattern generalization for function rules.

As we explored the kinds of geometric pattern tasks that lend themselves to figural reasoning, we found several resources that provide a variety of different tasks. In many instances, they also discuss ways in which these tasks might be analyzed and how they might lead to pattern generalizations for function rules. The analyses varied in their levels of abstraction and in how students were supported in their development of function rules. Although we did not find any discussion of a framework for characterizing the pattern-task complexity, we did find several examples that could be incorporated into a well-developed problem-solving process (Lee and Freiman 2006).

In this article, we will address several issues:

- First, we will look at a problem-solving process that supports the use of figural reasoning to explore and interpret geometric pattern tasks and generalize function rules.
- Second, we will discuss a framework for characterizing the complexities of geometric pattern tasks that might be used as applied contexts for figural reasoning.
- Third, we will summarize other considerations about how the long-term and extended use of geometric pattern tasks contributes to an overall development of students' functional thinking; such considerations are also important when developing a framework.

Pedagogy for Approaching These Tasks

In thinking about the framework, an additional component is the pedagogy that may be used to interact with students as they solve such problems. One common strategy asks students to begin with the first three or four stages in the sequence and use the three phases in **table 1** to guide investigation. Another strategy (Friel et al. 2009) involves providing only a third or fourth stage in a geometric pattern sequence and asking students to draw the missing early stages in the sequence to promote backward and forward thinking about possible relationships in the pattern.

An interesting variation of a starting-point task is to use a figure in a sequence that does not have an obvious designated place in a sequence. Its structure can be analyzed figurally without needing to reference a stage number. Boaler and Humphreys (2005) provide an excellent model using both text and video for a 10×10 image from the sequence in **figure 10**. See also a discussion that addresses the use of this task across grade levels, indicating the various ways that different grade levels of students might approach solutions (Ferrini-Mundy, Lappan, and Phillips 1997).

A PROBLEM-SOLVING PROCESS THAT PROMOTES FIGURAL REASONING

Friel, Rachlin, and Doyle (2001) provide guidelines for generating and describing growing sequences that are introduced through a variety of contexts that frequently involve geometric pattern tasks. One guideline that highlights figural reasoning involves "describing the shapes succinctly with words in such a way that someone who has not seen them will be able to duplicate the sequence" (p. 7).

Although there is an emphasis on figural reasoning, in most instances, the focus of analysis moves very

Table 1 These phases of the problem-solving process help in analyzing geometric pattern tasks (adapted from Lee and Freiman 2006).

Phase 1: *Reasoning figurally* using the visual characteristics of the geometric pattern task	1. How many different patterns can you see in this drawing? (See **fig. 17**.) a. How would you draw the next stage? b. How would you draw the 10th stage? c. How would you draw the 58th stage? d. How would you tell someone how to draw any stage at all? 2. I have a box of 25 smiley faces. How big a figure could I make? Would I have some smiley faces left over?
Phase 2: *Developing numerical relationships* to generalize a function	3. How many smiley faces does it take to make the 10th stage, the 58th stage, or the 100th stage? 4. How many smiley faces does it take to make the *n*th stage? 5. Which of the expressions for the *n*th stage is a "right" one?
Phase 3: *Extending pattern analysis*	6. Which stage has exactly 100 smiley faces in it? What about 50 smiley faces? 7. Can you create a pattern problem for the class?

Table 2 Three different figural reasoning strategies used to describe a single geometric pattern

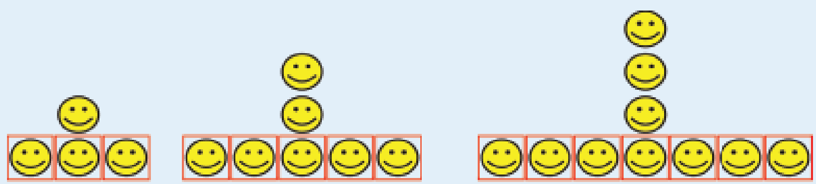
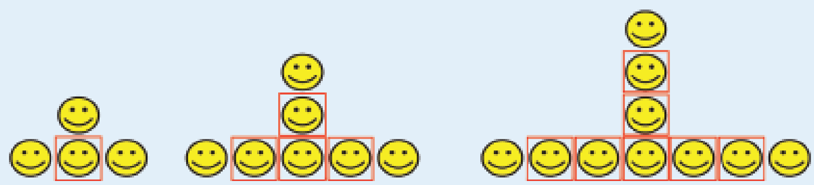
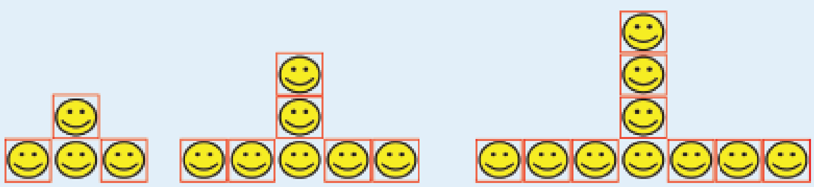
Fig. A 	This strategy focuses on the horizontal row of smiley faces with a column up from the middle. Drawing the 43rd stage would involve a horizontal row of two groups of 43 smiley faces plus 1 more and a stack 43 vertical smiley faces.
Fig. B 	This strategy involves using the previous figure and adding 1 more to each of three "arms." To find the 43rd stage would require knowing what the 42nd figure looked like (how many smiley faces) and then adding 3 more.
Fig. C 	This strategy highlights multiple sets of the same number of smiley faces. In the 43rd stage, there would be 3 sets of 43 smiley faces, with 1 smiley face in the middle.

Table 3 Numerical summaries of figural reasoning strategies

Stage	Reasoning	Total Smiley Faces	Explanation
Figure A			
1	$(1(h) + 1(h) + 1) + 1(v)$	4	The numerical statement of the strategy mirrors the student's thinking. This organization highlights what changes and what is constant. It makes an explicit connection between the stage number (input) and the total smiley faces (output).
2	$(2(h) + 2(h) + 1) + 2(v)$	7	
3	$(3(h) + 3(h) + 1) + 3(v)$	10	
4	$(4(h) + 4(h) + 1) + 4(v)$	13	
.	.	.	
.	.	.	
.	.	.	
n	$(n\,(h) + n\,(h) + 1) + n\,(v)$	$(n + n + 1) + n$	
Figure B			
1	$1 + 3$	4	Students may notice by this point that this pattern perception does not translate to numerical statements that make relationships as clear as with the other two strategies.
2	$4 + 3$	7	
3	$7 + 3$	10	
4	$10 + 3$	13	
.	.	.	
.	$? + 3$	?	
.	.	.	
n	.	.	
Figure C			
1	$1 + 1 + 1 + 1$	4	This reasoning leads to a function rule, again, that connects the input and the output. Students can compare the expression for figure A with that for figure C and consider the question, "Which of the expressions for the nth shape is a 'right' one?" Students can look back to the reasoning that produced each to determine if the expressions are equivalent.
2	$2 + 2 + 2 + 1$	7	
3	$3 + 3 + 3 + 1$	10	
4	$4 + 4 + 4 + 1$	13	
.	.	.	
.	.	.	
.	.	.	
n	$n + n + n + 1$	$3n + 1$	

(*Note:* h refers to horizontal parts, and v refers to vertical parts of the figure structure.)

quickly to how-many questions. For example, consider the geometric pattern in **figure 17**. Typical how-many questions include the following:

- How many smiley faces does it take to make the 10th stage?
- How many smiley faces does it take to make the 43rd stage?
- How many smiley faces does it take to make the 100th stage?
- How many smiley faces does it take to make the nth stage?

Often, steps that focus on the figural reasoning component of the task are neglected.

Lee and Freiman (2006) propose a set of questions to guide the problem-solving process that serves to highlight the visual reasoning phase, the cornerstone of figural reasoning. For example, for the smiley-face context described above, see the questions in **table 1**. If we focus on phase 1, students may visualize this geometric pattern in several different ways (Lee and Freiman 2006). Three possible strategies and their usefulness in promoting generalizations are discussed in **table 2**.

Students can talk through their figural-reasoning strategies and develop rules to match their patterns of thinking. After they have mastered these items, they can be introduced to the use of tables to record the numerical summaries. Visual strategies can be translated to numerical summaries, preserving a record of the thinking process, by using three-column tables (e.g., Lawrence and Hennessy 2002; Wickett, Kharas, and Burns 2002). Three such tables are presented in **table 3**, one for each of these strategies discussed in **table 2**.

When students problem solve and use a table with geometric pattern tasks, they are taking the first step into the world of functional thinking.

Next, we look at the characteristics of geometric pattern tasks that impact

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complexity. Any old pattern will not do; it is worth our time to think about when and how to introduce different types of patterns, while promoting functional thinking developed through figural reasoning.

ANALYZING PATTERN COMPLEXITY

Before you read on, take another look at the eighteen geometric patterns in **figures 1–18** on p. 26. Think about sorting the figures into two groups: simple patterns and more complex patterns.

- Which patterns would you consider simple? Why?
- Which patterns would you consider complex? Why?
- What are the similarities and differences among the patterns?
- How might these similarities and differences be highlighted to promote different figural reasoning strategies?

Although a teacher can ask various questions about any of these geometric patterns, consider queries that will help focus on figural reasoning strategies. As you look at each item, what different patterns do you see? How would you draw or build the next stage? The 10th stage? The 58th stage? How would you tell someone how to draw *any* stage in this pattern group? In our analysis of these and many other geometric patterns, in this phase of analysis we have identified several complexities that emerge and that deserve articulation.

The most basic geometric pattern reflects a linear, direct variation relationship, as shown in **figures 1** and **2**, in which the total number of blocks is a multiple of the stage number. Note that we refer to each figure in a sequence as a *stage number*. Other terms, such as *figure number*, *pattern number*, or *picture number*, may also be used. These numbers designate the order and sequence of the figures within a growing pattern and are the input to a function rule.

Once you move to phase 2 of the problem-solving process, calculating the total number of blocks involves a single, multiplicative step when relating the pattern number and the number of blocks. Adding a constant to a geometric pattern usually increases its complexity. Notice the difference between the patterns in **figures 1** and **3**. One additional tile is in each stage of the pattern in **figure 3**. This difference may seem minor, but it increases the complexity of the pattern by making the calculation of the total number of tiles (phase 2) a two-step process involving multiplication (by 1) and addition (plus 2). Thus, the functional relationship of the pattern in **figure 3** becomes $T = n + 1$, in which T is the total number of blocks, and n is the stage number.

For students who are just beginning to work with geometric patterns, identifying a constant may not be obvious. Figural-reasoning strategies can be prompted in different ways.

For example, the constant term can be represented using a different color. In **figures 3, 4,** and **7**, the constant (plus 1) is shown with a different-colored tile in each pattern sequence. Constants can also be shown using different shapes; the constants in **figures 5** and **6** (again, plus 1) are represented by a triangle at the top of the tree and a single center hexagon, respectively.

What makes the geometric patterns in **figures 5** and **11** more complex, however, is that two shapes grow in each successive stage. The tree in **figure 5** increases by both 1 square and 1 trapezoid in each stage. Students use figural reasoning when they observe these relationships. When students transition to phase 2 of the process, since there is n number of squares and n number of trapezoids, one way they can justify the functional relationship is through the rule $T = n + n + 1$, with the constant of 1 representing the triangle at the top.

The pattern in **figure 11** is more problematic. The number of hexagons is clear; this value corresponds to the stage number. However, although a student might expect the number of squares to grow by 6 in each stage (add 1 hexagon, so add 6 squares around it), this is not the case. One square can be pictured as overlapping, so that only 5 squares are in fact added in each successive stage.

The toothpicks illustration in

figure 9 presents a similar situation. Although a triangular shape is added in each successive stage, only 2 toothpicks are added to the stage, because a third toothpick would overlap with a toothpick already in place from the previous stage.

What makes the geometric pattern in **figure 12** more complex than that in **figure 7**? The difference between these two patterns is only in how they begin, but this makes its function more difficult to derive. In **figure 7**, the pattern has three spokes in each stage; the number of squares in each spoke corresponds to the stage number (see the student work in **fig. 19**). Thus, the functional relationship can be represented by the equation $T = 3n + 1$. In **figure 12**, the number of squares in each spoke actually corresponds to 1 fewer than the stage number. This functional relationship can be represented by the equation $T = 3(n - 1) + 1$. Students may have difficulty generating this rule if they are not encouraged to focus on the growing pattern itself and its translation to a numerical pattern through the use of a three-column table.

The geometric patterns in **figures 14** and **18** are more complex than the others because they represent nonlinear relationships. Although the pattern in **figure 14** looks exceedingly complex, it is actually well within grasp for students who have had prior experience with geometric patterns, especially in classrooms in which figural reasoning has been highlighted. There are multiple ways of seeing the pattern in **figure 14** (see Smith, Silver, and Stein 2005). In **figure 20**, students present a summary report of their analysis of **figure 14** with a table showing the pattern structure. In their summary, they highlight the components related to how they analyzed the pattern. This pattern, like many others, can promote a classroom

Fig. 19 Students' summary report of their analysis of **figure 7**, using the strategy of three identical extensions and a single center (yellow) square.

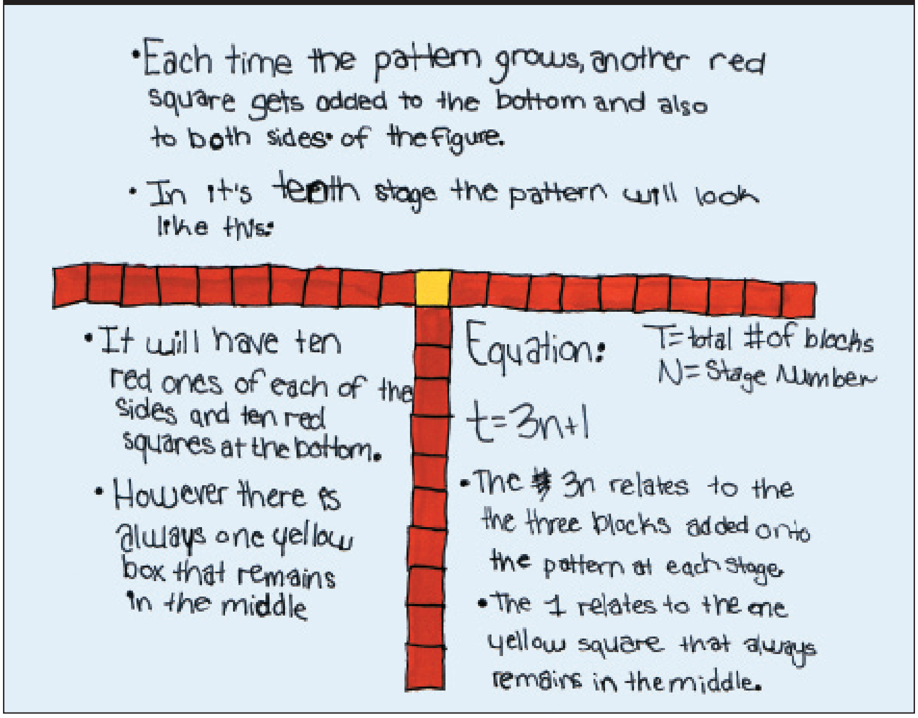

Fig. 20 Students' summary report of their analysis of **figure 14,** demonstrating their strategy of identifying a center $n \times n$ part, exterior parts, and two additional squares.

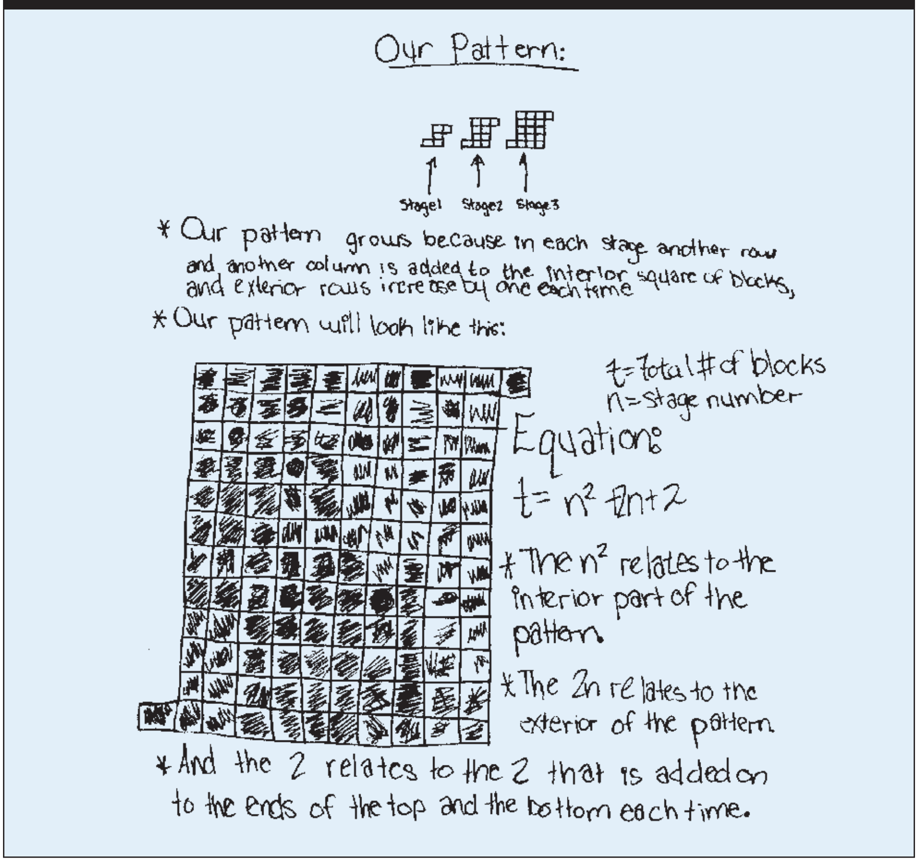

Table 4 Equation key

Figure Number	Possible Solution Equations
1	Tiles: $T = n$
2	Tiles: $T = 2n$
3	Tiles: $T = n + 1$
4	Tiles: $T = n + 4$
5	Triangles: $T = 1$ Trapezoids: $T = n$ Squares: $T = n$ Total pieces: $T = 2n + 1$
6	Hexagons: $T = 1$ Squares: $T = 6n$ Total pieces: $T = 6n + 1$
7	Tiles: $T = 3n + 1$
8	Perimeter: $T = 2n + 2$
9	Toothpicks: $T = 2n + 1$
10	Center tiles: $T = n^2$ Border tiles: $T = 4n + 4$ Total tiles: $T = n^2 + 4n + 4$
11	Hexagons: $T = n$ Squares: $T = 5n + 1$ Total pieces: $T = 6n + 1$
12	Tiles: $T = 3n - 2$
13	Black tiles: $T = 4n + 1$ White tiles: $T = 8n + 8$ Total tiles: $T = 12n + 9$
14	Tiles: $T = n^2 + 2n + 2$
15	Cubes: $T = 3n + 1$
16	Surface area-smiley faces: $T = 4n + 2$
17	Smiley faces: $T = 3n + 1$
18	Tiles: $T = n^2$

discussion of equivalence of symbolic expressions and multiple answers.

The pattern in **figure 18**, however, does not lend itself to analysis using figural reasoning. Sasman, Olivier, and Linchevski (cited in Rivera 2007) distinguish between transparent and nontransparent geometric patterns. The other seventeen patterns are transparent patterns; the functional relationships can be obtained easily through figural reasoning. However, **figure 18**'s pattern is nontransparent, in that "something more needs to be done before students are able to see a possible function rule from the available cues" (Rivera 2007, p. 72). The squares to the left of the tallest column could be detached, rotated 180 degrees, and fit into the staircase to the right, thereby creating a square.

The patterns in **figures 8, 15,** and **16** provide examples of how to add complexity to more simple patterns. The geometric pattern in **figure 8** is identical to **figure 1**. However, by asking a more challenging question, "How many people could sit around the table in each stage?" it becomes a contextual perimeter problem, with a more complex functional relationship.

Figure 16 brings this same pattern into three dimensions. The question, "How many smiley face stickers does it take to cover the figure?" can promote a rich connection to surface area. Its functional relationship is well within reach of students who are encouraged to articulate their process of figural reasoning. Likewise, you may have noticed that the pattern in **figure 15** is a three-dimensional version of the pattern in **figure 7**. This three-dimensional version allows for more complex questions involving volume and surface area.

Table 4 summarizes possible solution equations for each of the eighteen patterns presented. Of course, students could engage in any of these tasks using the framework discussed here; this table gives the reader explicit forms of pattern generalization (Friel et al. 2001, pp. 7-8), relating figure structures and stage numbers.

A combination of an effective problem-solving process that focuses on figural reasoning and appropriately challenging geometric pattern tasks will enable mathematics teachers at all levels to promote functional understanding. Although the variety of considerations of how these tasks may be used is beyond the scope of this article, some factors for incorporating these tasks across grade levels are discussed.

USING THE FRAMEWORK WITH INSTRUCTION

The increasing emphasis on algebraic thinking across grades K–12 requires that attention be paid to the use of figural reasoning. When coupled with an analysis of geometric patterns, students are on the path to developing solid functional thinking. With this scenario in mind, it is possible to begin to think about the kinds of tasks that might be used at grade 3 versus grade 8. Organizing a loose but reasoned trajectory of growing pattern tasks to be used across grades 2–8 will give teachers and students many opportunities to explore functional thinking in visual settings. One way is to have students investigate "families" of related pattern tasks.

For example, **figure 8** is a count-the-perimeter task (see Smith, Silver, and Stein 2005). What happens when students look at strings of triangles or pentagons or hexagons or other types of patterns composed of more than one kind of polygon? Another possibility is to align choices of pattern tasks that are organized in an instructional sequence and are used within and across multiple grade levels (Smith, Hillen, and Catania 2007). What alignment issues need to be addressed? How might a teacher in grade 8 build on the experiences from those in grade 7? More explicitly, if this alignment process is addressed consistently at all grade levels, how would students' functional thinking emerge and develop? The answer is possibly at a depth and in ways that we have yet to consider possible.

Using geometric pattern tasks to explore functional relationships has its limitations. Stage numbers are limited to positive numbers, the functional relationships are not continuous, and the contexts are necessarily restricted. However, we believe that incorporating such tasks into mathematics classrooms offers a valuable way to promote figural reasoning and develop a rich conceptual understanding of functions. The framework for exploring these tasks should provide a starting point for further analysis of these patterns and the multiple ways that they can be used to promote functional thinking.

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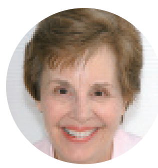

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